7.7 Fluid Pressure and Fluid Force

- **Find fluid pressure and fluid force.**

Fluid Pressure and Fluid Force

Swimmers know that the deeper an object is submerged in a fluid, the greater the pressure on the object. **Pressure** is defined as the force per unit of area over the surface of a body. For example, because a column of water that is 10 feet in height and 1 inch square weighs 4.3 pounds, the *fluid pressure* at a depth of 10 feet of water is 4.3 pounds per square inch.* At 20 feet, this would increase to 8.6 pounds per square inch, and in general the pressure is proportional to the depth of the object in the fluid.

BLAISE PASCAL (1623–1662)

Pascal is well known for his work in many areas of mathematics and physics, and also for his influence on Leibniz. Although much of Pascal's work in calculus was intuitive and lacked the rigor of modern mathematics, he nevertheless anticipated many important results.

See LarsonCalculus.com to read more of this biography.

Definition of Fluid Pressure

The **pressure** on an object at depth h in a liquid is

$$\text{Pressure} = P = wh$$

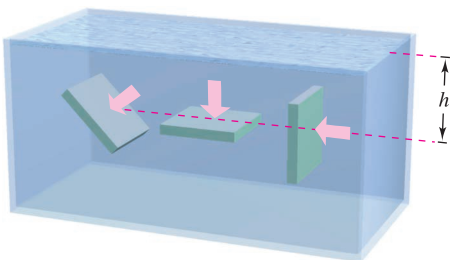
where w is the weight-density of the liquid per unit of volume.

Below are some common weight-densities of fluids in pounds per cubic foot.

Ethyl alcohol	49.4
Gasoline	41.0–43.0
Glycerin	78.6
Kerosene	51.2
Mercury	849.0
Seawater	64.0
Water	62.4

When calculating fluid pressure, you can use an important (and rather surprising) physical law called **Pascal's Principle,** named after the French mathematician Blaise Pascal. Pascal's Principle states that the pressure exerted by a fluid at a depth h is transmitted equally *in all directions.* For example, in Figure 7.65, the pressure at the indicated depth is the same for all three objects. Because fluid pressure is given in terms of force per unit area ($P = F/A$), the fluid force on a *submerged horizontal* surface of area A is

$$\text{Fluid force} = F = PA = (\text{pressure})(\text{area}).$$



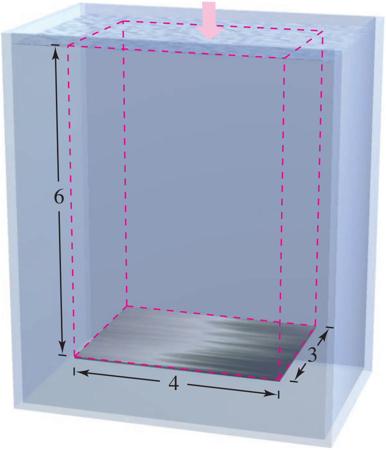
The pressure at h is the same for all three objects.
Figure 7.65

* The total pressure on an object in 10 feet of water would also include the pressure due to Earth's atmosphere. At sea level, atmospheric pressure is approximately 14.7 pounds per square inch.

The Granger Collection, New York

EXAMPLE 1 Fluid Force on a Submerged Sheet

Find the fluid force on a rectangular metal sheet measuring 3 feet by 4 feet that is submerged in 6 feet of water, as shown in Figure 7.66.



The fluid force on a horizontal metal sheet is equal to the fluid pressure times the area.
Figure 7.66

Solution Because the weight-density of water is 62.4 pounds per cubic foot and the sheet is submerged in 6 feet of water, the fluid pressure is

$$P = (62.4)(6) \qquad P = wh$$
$$= 374.4 \text{ pounds per square foot.}$$

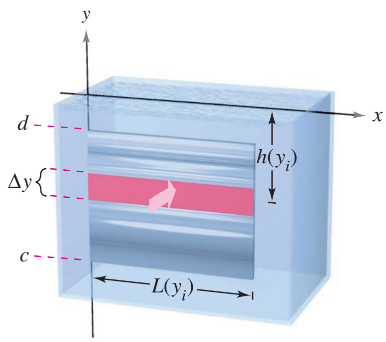
Because the total area of the sheet is $A = (3)(4) = 12$ square feet, the fluid force is

$$F = PA$$
$$= \left(374.4 \frac{\text{pounds}}{\text{square foot}}\right)(12 \text{ square feet})$$
$$= 4492.8 \text{ pounds.}$$

This result is independent of the size of the body of water. The fluid force would be the same in a swimming pool or lake.

In Example 1, the fact that the sheet is rectangular and horizontal means that you do not need the methods of calculus to solve the problem. Consider a surface that is submerged vertically in a fluid. This problem is more difficult because the pressure is not constant over the surface.

Consider a vertical plate that is submerged in a fluid of weight-density w (per unit of volume), as shown in Figure 7.67. To determine the total force against *one side* of the region from depth c to depth d, you can subdivide the interval $[c, d]$ into n subintervals, each of width Δy. Next, consider the representative rectangle of width Δy and length $L(y_i)$, where y_i is in the ith subinterval. The force against this representative rectangle is



Calculus methods must be used to find the fluid force on a vertical metal plate.
Figure 7.67

$$\Delta F_i = w(\text{depth})(\text{area})$$
$$= wh(y_i)L(y_i)\,\Delta y.$$

The force against n such rectangles is

$$\sum_{i=1}^{n} \Delta F_i = w \sum_{i=1}^{n} h(y_i)L(y_i)\,\Delta y.$$

Note that w is considered to be constant and is factored out of the summation. Therefore, taking the limit as $\|\Delta\| \to 0$ $(n \to \infty)$ suggests the next definition.

Definition of Force Exerted by a Fluid

The **force F exerted by a fluid** of constant weight-density w (per unit of volume) against a submerged vertical plane region from $y = c$ to $y = d$ is

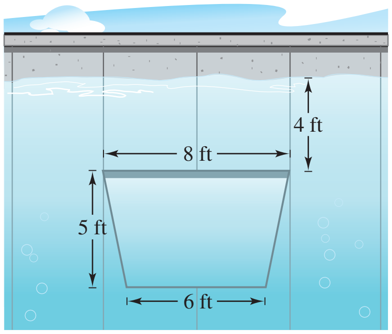
$$F = w \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} h(y_i)L(y_i)\,\Delta y$$
$$= w \int_c^d h(y)L(y)\,dy$$

where $h(y)$ is the depth of the fluid at y and $L(y)$ is the horizontal length of the region at y.

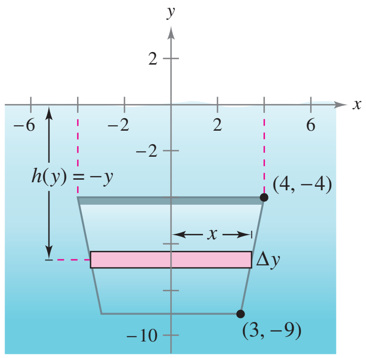
EXAMPLE 2 Fluid Force on a Vertical Surface

See LarsonCalculus.com for an interactive version of this type of example.

A vertical gate in a dam has the shape of an isosceles trapezoid 8 feet across the top and 6 feet across the bottom, with a height of 5 feet, as shown in Figure 7.68(a). What is the fluid force on the gate when the top of the gate is 4 feet below the surface of the water?



(a) Water gate in a dam



(b) The fluid force against the gate

Figure 7.68

Solution In setting up a mathematical model for this problem, you are at liberty to locate the x- and y-axes in several different ways. A convenient approach is to let the y-axis bisect the gate and place the x-axis at the surface of the water, as shown in Figure 7.68(b). So, the depth of the water at y in feet is

$$\text{Depth} = h(y) = -y.$$

To find the length $L(y)$ of the region at y, find the equation of the line forming the right side of the gate. Because this line passes through the points $(3, -9)$ and $(4, -4)$, its equation is

$$\begin{aligned} y - (-9) &= \frac{-4 - (-9)}{4 - 3}(x - 3) \\ y + 9 &= 5(x - 3) \\ y &= 5x - 24 \\ x &= \frac{y + 24}{5}. \end{aligned}$$

In Figure 7.68(b) you can see that the length of the region at y is

$$\text{Length} = 2x = \frac{2}{5}(y + 24) = L(y).$$

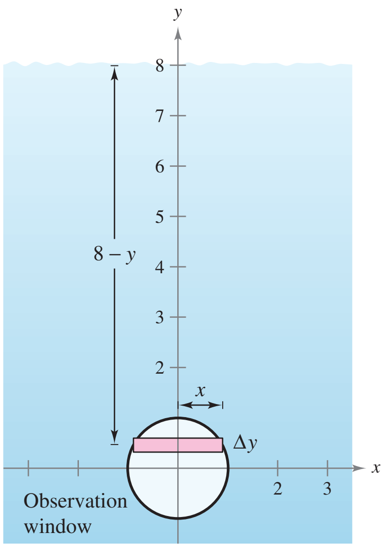
Finally, by integrating from $y = -9$ to $y = -4$, you can calculate the fluid force to be

$$\begin{aligned} F &= w\int_c^d h(y)L(y)\,dy \\ &= 62.4\int_{-9}^{-4} (-y)\left(\frac{2}{5}\right)(y + 24)\,dy \\ &= -62.4\left(\frac{2}{5}\right)\int_{-9}^{-4} (y^2 + 24y)\,dy \\ &= -62.4\left(\frac{2}{5}\right)\left[\frac{y^3}{3} + 12y^2\right]_{-9}^{-4} \\ &= -62.4\left(\frac{2}{5}\right)\left(\frac{-1675}{3}\right) \\ &= 13{,}936 \text{ pounds.} \end{aligned}$$

In Example 2, the x-axis coincided with the surface of the water. This was convenient, but arbitrary. In choosing a coordinate system to represent a physical situation, you should consider various possibilities. Often you can simplify the calculations in a problem by locating the coordinate system to take advantage of special characteristics of the problem, such as symmetry.

EXAMPLE 3 Fluid Force on a Vertical Surface

A circular observation window on a marine science ship has a radius of 1 foot, and the center of the window is 8 feet below water level, as shown in Figure 7.69. What is the fluid force on the window?



The fluid force on the window
Figure 7.69

Solution To take advantage of symmetry, locate a coordinate system such that the origin coincides with the center of the window, as shown in Figure 7.69. The depth at y is then

$$\text{Depth} = h(y) = 8 - y.$$

The horizontal length of the window is $2x$, and you can use the equation for the circle, $x^2 + y^2 = 1$, to solve for x as shown.

$$\begin{aligned}\text{Length} &= 2x\\ &= 2\sqrt{1-y^2} = L(y)\end{aligned}$$

Finally, because y ranges from -1 to 1, and using 64 pounds per cubic foot as the weight-density of seawater, you have

$$\begin{aligned}F &= w\int_c^d h(y)L(y)\,dy\\ &= 64\int_{-1}^{1}(8-y)(2)\sqrt{1-y^2}\,dy.\end{aligned}$$

Initially it looks as though this integral would be difficult to solve. However, when you break the integral into two parts and apply symmetry, the solution is simpler.

$$F = 64(16)\int_{-1}^{1}\sqrt{1-y^2}\,dy - 64(2)\int_{-1}^{1}y\sqrt{1-y^2}\,dy$$

The second integral is 0 (because the integrand is odd and the limits of integration are symmetric with respect to the origin). Moreover, by recognizing that the first integral represents the area of a semicircle of radius 1, you obtain

$$\begin{aligned}F &= 64(16)\left(\frac{\pi}{2}\right) - 64(2)(0)\\ &= 512\pi\\ &\approx 1608.5 \text{ pounds.}\end{aligned}$$

So, the fluid force on the window is about 1608.5 pounds.

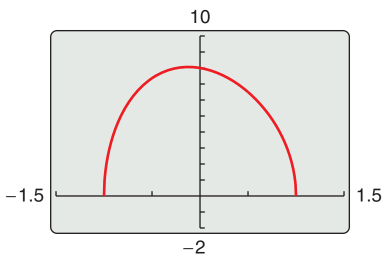
TECHNOLOGY To confirm the result obtained in Example 3, you might have considered using Simpson's Rule to approximate the value of

$$128\int_{-1}^{1}(8-x)\sqrt{1-x^2}\,dx.$$

From the graph of

$$f(x) = (8-x)\sqrt{1-x^2}$$

however, you can see that f is not differentiable when $x = \pm 1$ (see figure at the right). This means that you cannot apply Theorem 4.20 from Section 4.6 to determine the potential error in Simpson's Rule. Without knowing the potential error, the approximation is of little value. Use a graphing utility to approximate the integral.



f is not differentiable at $x = \pm 1$.

7.7 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Force on a Submerged Sheet **In Exercises 1–4, the area of the top side of a piece of sheet metal is given. The sheet metal is submerged horizontally in 8 feet of water. Find the fluid force on the top side.**

1. 3 square feet

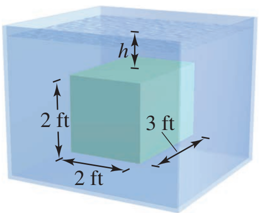
2. 8 square feet

3. 10 square feet

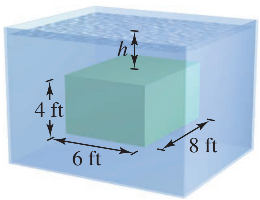
4. 25 square feet

Buoyant Force **In Exercises 5 and 6, find the buoyant force of a rectangular solid of the given dimensions submerged in water so that the top side is parallel to the surface of the water. The buoyant force is the difference between the fluid forces on the top and bottom sides of the solid.**

5.

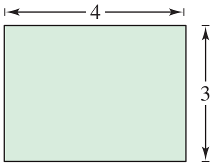


6.

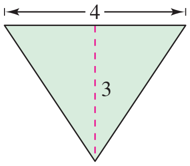


Fluid Force on a Tank Wall **In Exercises 7–12, find the fluid force on the vertical side of the tank, where the dimensions are given in feet. Assume that the tank is full of water.**

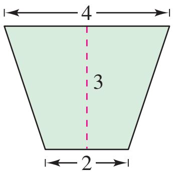
7. Rectangle



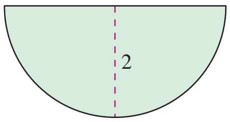
8. Triangle



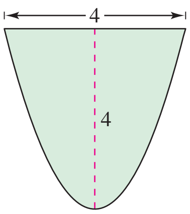
9. Trapezoid



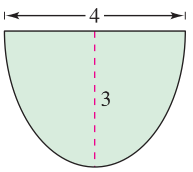
10. Semicircle



11. Parabola, $y = x^2$

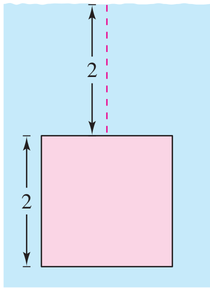


12. Semiellipse, $y = -\frac{1}{2}\sqrt{36 - 9x^2}$

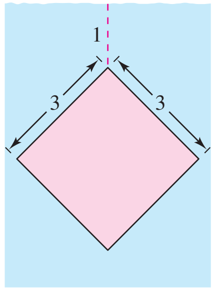


Fluid Force of Water **In Exercises 13–16, find the fluid force on the vertical plate submerged in water, where the dimensions are given in meters and the weight-density of water is 9800 newtons per cubic meter.**

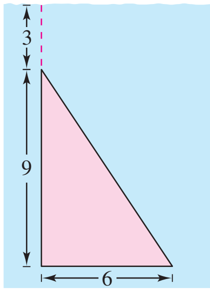
13. Square



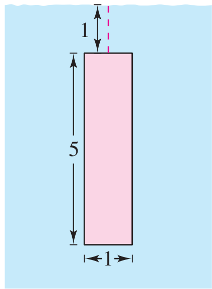
14. Square



15. Triangle

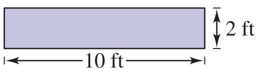


16. Rectangle

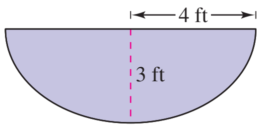


Force on a Concrete Form **In Exercises 17–20, the figure is the vertical side of a form for poured concrete that weighs 140.7 pounds per cubic foot. Determine the force on this part of the concrete form.**

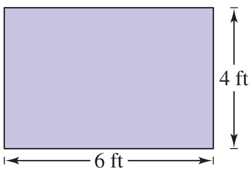
17. Rectangle



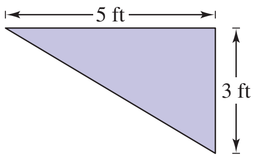
18. Semiellipse, $y = -\frac{3}{4}\sqrt{16 - x^2}$



19. Rectangle



20. Triangle



21. Fluid Force of Gasoline A cylindrical gasoline tank is placed so that the axis of the cylinder is horizontal. Find the fluid force on a circular end of the tank when the tank is half full, where the diameter is 3 feet and the gasoline weighs 42 pounds per cubic foot.

22. Fluid Force of Gasoline Repeat Exercise 21 for a tank that is full. (Evaluate one integral by a geometric formula and the other by observing that the integrand is an odd function.)

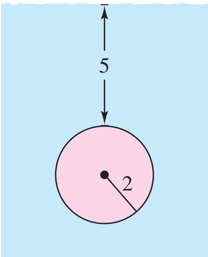
23. Fluid Force on a Circular Plate A circular plate of radius r feet is submerged vertically in a tank of fluid that weighs w pounds per cubic foot. The center of the circle is k feet below the surface of the fluid, where $k > r$. Show that the fluid force on the surface of the plate is

$$F = wk(\pi r^2).$$

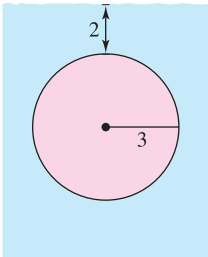
(Evaluate one integral by a geometric formula and the other by observing that the integrand is an odd function.)

24. Fluid Force on a Circular Plate Use the result of Exercise 23 to find the fluid force on the circular plate shown in each figure. Assume the plates are in the wall of a tank filled with water and the measurements are given in feet.

(a)



(b)

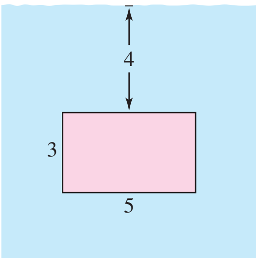


25. Fluid Force on a Rectangular Plate A rectangular plate of height h feet and base b feet is submerged vertically in a tank of fluid that weighs w pounds per cubic foot. The center is k feet below the surface of the fluid, where $k > h/2$. Show that the fluid force on the surface of the plate is

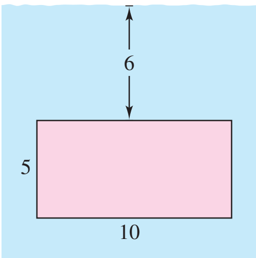
$$F = wkhb.$$

26. Fluid Force on a Rectangular Plate Use the result of Exercise 25 to find the fluid force on the rectangular plate shown in each figure. Assume the plates are in the wall of a tank filled with water and the measurements are given in feet.

(a)



(b)

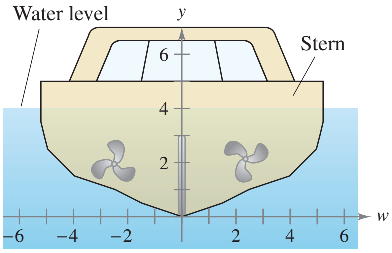


27. Submarine Porthole A square porthole on a vertical side of a submarine (submerged in seawater) has an area of 1 square foot. Find the fluid force on the porthole, assuming that the center of the square is 15 feet below the surface.

28. Submarine Porthole Repeat Exercise 27 for a circular porthole that has a diameter of 1 foot. The center is 15 feet below the surface.

29. Modeling Data The vertical stern of a boat with a superimposed coordinate system is shown in the figure. The table shows the widths w of the stern (in feet) at indicated values of y. Find the fluid force against the stern.

y	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
w	0	3	5	8	9	10	10.25	10.5	10.5



30. Irrigation Canal Gate The vertical cross section of an irrigation canal is modeled by $f(x) = 5x^2/(x^2 + 4)$, where x is measured in feet and $x = 0$ corresponds to the center of the canal. Use the integration capabilities of a graphing utility to approximate the fluid force against a vertical gate used to stop the flow of water when the water is 3 feet deep.

WRITING ABOUT CONCEPTS

31. Think About It Approximate the depth of the water in the tank in Exercise 7 if the fluid force is one-half as great as when the tank is full. Explain why the answer is not $\frac{3}{2}$.

32. Fluid Pressure and Fluid Force

(a) Define fluid pressure.

(b) Define fluid force against a submerged vertical plane region.

33. Fluid Pressure Explain why fluid pressure on a surface is calculated using horizontal representative rectangles instead of vertical representative rectangles.

34. HOW DO YOU SEE IT? Two identical semicircular windows are placed at the same depth in the vertical wall of an aquarium (see figure). Which is subjected to the greater fluid force? Explain.